Math 1B – Calculus II – Chapter 6 Test – Fall ’08 Name________________________

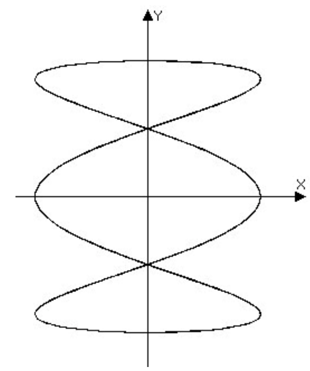
Write all responses on separate paper. Show your work for credit.

1. Consider the region $\mathscr{R}$ in the first quadrant bounded by the y-axis and the curves $y = 2\cos x$ and $y = \sin x$.
   Set up (but **do not evaluate**) integrals to compute the following:
   a. The area of $\mathscr{R}$ by integrating over x.
   b. The area of $\mathscr{R}$ by integrating over y.
   c. The volume of the region generated by rotating $\mathscr{R}$ about the y-axis.
   d. The volume of the region generated by rotating $\mathscr{R}$ about the x-axis.
   e. The surface area of the volume generated by revolving $\mathscr{R}$ about the line $x = \arctan(2)$.

2. Consider the parametric equations

$$x = \cos(3t)$$
$$y = \sin(t)$$

   for the lissajous figure whose graph is shown at right.



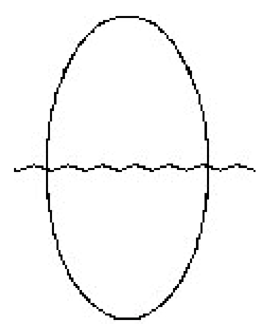
   a. Set up (but **do not evaluate**) an integral for the arc length of this curve.

   b. Set up (but **do not evaluate**) an integral for the volume generated by revolving this curve about the y-axis.

3. Suppose a pyramid with a square base of area 225 square meters and a height of 160 meters is filled with water. Find the work required to pump all the water out of the top of a pyramid.

4. Consider the function $f(x) = x \cdot e^{-x}$ on the interval [0,ln2].
   a. Find the average value of f on the interval.
   b. Why does this function satisfy the condition of the Mean Value Theorem for Integrals?
   c. What equation would you solve to find the number whose existence is guaranteed by the Mean Value Theorem for Integrals for this function on that interval? Can you solve it exactly?

5. Suppose an elliptical plate is with width = 2 meters and height = 4 meters is half submerged in water as shown at right. Express the force against one side of the plate as an integral. Do not evaluate the integral.



   Recall that the equation for an ellipse with half width a and half-height b and centered at (h, k) is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

6. Find the centroid of the region $\mathscr{R}$ described in problem #1.

7. Consider $f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{2}\right) & \text{if} \quad -1 \le x \le 1 \\ 0 & \text{if} \quad x < -1 \text{ or } x > 1 \end{cases}$
   a. Explain why f is a probability density function.
   b. Find $P\left(X < \dfrac{1}{2}\right)$
   c. Calculate the mean. Is the value what you expect?

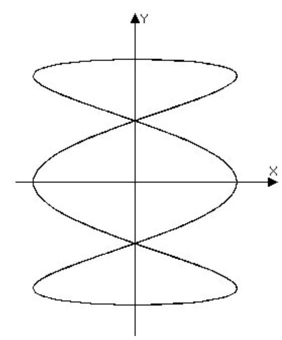
Bonus problem:

8. The bottom of a small shallow circular pond has the shape of the parabola $y = (x/10)^2$, rotated around the y-axis (units: meters). Its radius is 10 meters; its depth in the center is one meter. Algae are in the pond, but not spread uniformly. Set up (but **do not evaluate**) two definite integrals which give the total mass of algae in the pond if the concentration C of algae (in g/m^3) at each point P in the pond is as given below (two cases):
   a. C is numerically equal to the depth (in meters) of the point P.
   b. C is numerically equal to $\frac{1}{1+r^2}$, where r is the horizontal distance of P from the center of the pond.

Math 1B – Calculus II – Chapter 6 Test Solutions – Fall '08

1. Consider the region $\mathscr{R}$ in the first quadrant bounded by the y-axis and the curves $y = 2\cos x$ and $y = \sin x$.
   Set up (but **do not evaluate**) integrals to compute the following:
   a. The area of $\mathscr{R}$ by integrating over x.
      SOLN: $\int_0^{\tan^{-1}2}(2\cos x - \sin x)\,dx$
   b. The area of $\mathscr{R}$ by integrating over y.
      SOLN: $\int_0^{2/\sqrt{5}} \sin^{-1} y dy + \int_{2/\sqrt{5}}^{2} \cos^{-1}\frac{y}{2} dy$
   c. The volume of the region generated by rotating $\mathscr{R}$ about the y-axis.
      SOLN: Using shells: $2\pi\int_0^{\tan^{-1}2} x(2\cos x - \sin x)\,dx$
      With washers: $\pi\left[\int_0^{2/\sqrt{5}}\left(\sin^{-1} y\right)^2 dy + \int_{2/\sqrt{5}}^{2}\left(\cos^{-1}\frac{y}{2}\right)^2 dy\right]$
   d. The volume of the region generated by rotating $\mathscr{R}$ about the x-axis.
      SOLN: Using washers: $\pi\int_0^{\tan^{-1}2}(2\cos x)^2 - (\sin x)^2\,dx$
      With shells: $2\pi\left[\int_0^{2/\sqrt{5}} y\left(\sin^{-1} y\right)dy + \int_{2/\sqrt{5}}^{2} y\left(\cos^{-1}\frac{y}{2}\right)dy\right]$
   e. The surface area of the volume generated by revolving $\mathscr{R}$ about the line $x = \arctan(2)$.
      SOLN: The outside of the cylinder is $4\pi\arctan(2)$. For the inside you need an integral:
      $2\pi\int_0^{\arctan 2} x\left(\sqrt{1+\cos^2 x} + \sqrt{1+4\sin^2 x}\right)dx$

2. Consider the parametric equations
   $x = \cos(3t)$
   $y = \sin(t)$
   for the lissajous figure whose graph is shown at right.



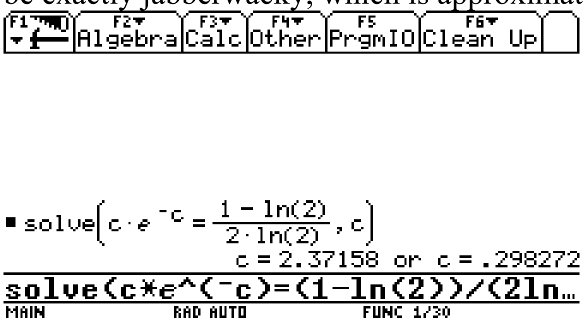
   a. Set up (but **do not evaluate**) an integral for the arc length of this curve.
      SOLN: $4\int_0^{\pi/2}\sqrt{9\sin^2(3t)+\cos^2 t}\,dt$
   b. Set up (but **do not evaluate**) an integral for the volume generated by revolving this curve about the y-axis.
      SOLN: $2\pi\int_0^{\pi/2}\cos^2 3t\cos t\,dt$

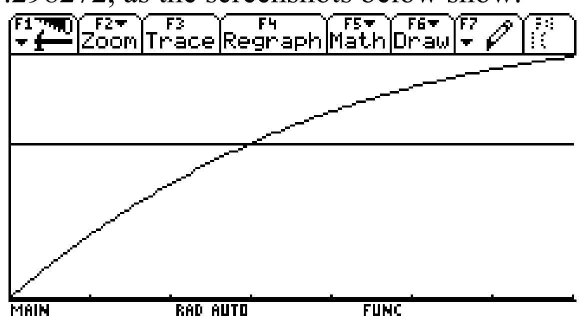
3. Suppose a pyramid with a square base of area 225 square meters and a height of 160 meters is filled with water. Find the work required to pump all the water out of the top of a pyramid.
   SOLN: Whoops: I guess the true dimension of the great pyramid of Egypt is more like the length of an edge of the square base is 225 meters. But we solve the problems we are given, so here it goes:
   The horizontal cross-sections are squares whose side lengths varies linearly from 15 to 0 as the height y ranges from 0 to 160. Thus the cross sectional area at height y is $A(y) = \left(15 - \frac{15y}{160}\right)^2$ - or – if you think of x as the distance from the top, then the cross-sectional area at x is $A(x) = \left(\frac{15x}{160}\right)^2 = \frac{9x^2}{1024}$. This is maybe easier to work with. Then an element of work is

$dW = (\text{dist})\,dF = x(9810)\,dV = 9810\left(\frac{9x^3}{1024}\right)dx$. Integrating gives the total amount of work:

$$\int_0^{160} 9810\left(\frac{9x^3}{1024}\right)dx = \frac{4905(9)}{2048}x^4\Big|_0^{160} = \frac{5^5\cdot 3^2\cdot 109\cdot 2^{20}}{2^{11}} = 3^2\cdot 109\cdot 2^4\cdot 10^5 = 109\cdot 144\cdot 10^5 = 1569600000\,\text{J}$$

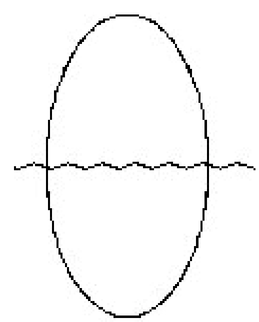
4. Consider the function $f(x) = x\cdot e^{-x}$ on the interval [0,ln2].
   a. Find the average value of *f* on the interval. SOLN:
   $$f_{\text{AVG}} = \frac{1}{\ln 2}\int_0^{\ln 2} x\cdot e^{-x}dx = \frac{1}{\ln 2}\left(-xe^{-x}\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-x}dx\right) = \frac{1}{\ln 2}\left(-\frac{\ln 2}{2} - \left(\frac{1}{2}-1\right)\right) = \frac{1-\ln 2}{2\ln 2} \approx 0.2213$$
   b. Why does this function satisfy the condition of the Mean Value Theorem for Integrals?
   SOLN: $f(x) = x\cdot e^{-x}$ is continuous on the interval [0,ln2].
   c. What equation would you solve to find the number whose existence is guaranteed by the Mean Value Theorem for Integrals for this function on that interval? Can you solve it exactly?
   SOLN: $c\cdot e^{-c} = \frac{1-\ln 2}{2\ln 2}$ which doesn't a have a nice closed form solution, so it hard to solve it "exactly." We could give the solution a name, like, say, "jabberwacky" and then the solution would be exactly jabberwacky, which is approximately 0.298272, as the screenshots below show.





5. Suppose an elliptical plate is with width = 2 meters and height = 4 meters is half submerged in water as shown at right. Express the force against one side of the plate as an integral. Do not evaluate the integral.

   Recall that the equation for an ellipse with half width *a* and half-height *b* and centered at (*h*, *k*) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
   SOLN: Take the center of the ellipse as the origin and write the equation for the ellipse: $x^2 + \frac{y^2}{4} = 1$ so that for a given depth, –*y* we have a cross-sectional length $2x = \sqrt{4-y^2}$ whence the total force is
   $$9810\int_{-2}^{0} -y\sqrt{4-y^2}\,dy = 4905\int_0^4 \sqrt{u}\,du = 3270(4)^{3/2} = 25160\,\text{Newtons}$$



6. Find the centroid of the region $\mathscr{R}$ described in problem #1.
   SOLN: The area is $\int_0^{\tan^{-1}2}(2\cos x - \sin x)\,dx = 2\sin(\tan^{-1}2) + \cos(\tan^{-1}2) - 1 = \sqrt{5}-1$
   The moment about the *y*-axis is

$$M_y = \int_0^{\tan^{-1}2} x(2\cos x - \sin x)\,dx$$

$$= x(2\sin x + \cos x)\Big|_0^{\tan^{-1}2} - \int_0^{\tan^{-1}2} 2\sin x + \cos x dx = \sqrt{5}\tan^{-1}2 - (-2\cos x + \sin x)\Big|_0^{\tan^{-1}2} = 2 + \sqrt{5}\tan^{-1}2$$

The moment about the x axis is

$$M_x = \frac{1}{2}\int_0^{\tan^{-1}2} (2\cos x)^2 - \sin^2 x dx = \frac{1}{2}\int_0^{\tan^{-1}2} 4 - 5\sin^2 x dx = \int_0^{\tan^{-1}2} \frac{3+5\cos 2x}{4} dx$$

$$= \frac{3}{4}\tan^{-1}2 + \frac{5}{4}\sin\left(\tan^{-1}2\right)\cos\left(\tan^{-1}2\right) = \frac{3}{4}\tan^{-1}2 + \frac{1}{2}$$

Thus the centroid is $(\bar{x},\bar{y}) = \left(\frac{2+\sqrt{5}\tan^{-1}2}{\sqrt{5}-1}, \frac{2+3\tan^{-1}2}{4\sqrt{5}-4}\right) \approx (0.3848, 1.076)$

7. Consider $f(x) = \begin{cases} \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right) & \text{if} \quad -1 \le x \le 1 \\ 0 & \text{if} \quad x < -1 \text{ or } x > 1 \end{cases}$
   a. Explain why f is a probability density function.
      SOLN: First, f is non-negative on the interval $[-1,1]$ since this corresponds to the 1st and 4th quadrants of the unit circle, where the x-coordinate (the cosine function) is non-negative.
      Then, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} \frac{\pi}{4}\cos\left(\frac{\pi x}{2}\right)dx = \frac{1}{2}\sin\left(\frac{\pi x}{2}\right)\Big|_{-1}^{1} = \frac{1}{2}\left(1-(-1)\right) = 1$, so it's a p. density function.
   b. Find $P\left(X < \frac{1}{2}\right)$
      SOLN: $\frac{\pi}{4}\int_{-1}^{1/2} \cos\left(\frac{\pi x}{2}\right)dx = \frac{1}{2}\sin\left(\frac{\pi x}{2}\right)\Big|_{-1}^{1/2} = \frac{1}{2}\left(\frac{\sqrt{2}}{2} - (-1)\right) = \frac{\sqrt{2}+2}{4} \approx 0.8536$
   c. Calculate the mean. Is the value what you expect?
      SOLN: $\frac{\pi}{4}\int_{-1}^{1} x\cos\left(\frac{\pi x}{2}\right)dx = \frac{x}{2}\sin\left(\frac{\pi x}{2}\right)\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1}\sin\left(\frac{\pi x}{2}\right)dx = \left(\frac{1}{2}-\frac{1}{2}\right) + \pi\cos\left(\frac{\pi x}{2}\right)\Big|_{-1}^{1} = 0$
      This is reasonable since the area is symmetric about the y axis.

Bonus problem:

8. The bottom of a small shallow circular pond has the shape of the parabola $y = (x/10)^2$, rotated around the y-axis (units: meters). Its radius is 10 meters; its depth in the center is one meter. Algae are in the pond, but not spread uniformly. Set up (but **do not evaluate**) two definite integrals which give the total mass of algae in the pond if the concentration C of algae (in g/m^3) at each point P in the pond is as given below (two cases):
   a. C is numerically equal to the depth (in meters) of the point P.
      SOLN: element of mass = (mass density)*(element of volume) = $(1-y)\pi x^2 dy = 100(1-y)\pi y dy$ so the total mass is $100\int_0^1 (1-y)\,ydy$
   b. C is numerically equal to $\frac{1}{1+r^2}$, where r is the horizontal distance of P from the center of the pond.
      element of mass = (mass density)*(element of volume) = $\frac{1}{1+x^2}2\pi x(1-y)\,dx = \frac{2\pi x}{1+x^2}\left(1-\frac{x^2}{100}\right)dx$
      so the total mass is $\int_0^{10} \frac{2\pi x}{1+x^2}\left(1-\frac{x^2}{100}\right)dx$